AIAA 78-1439R

The Main Problem in the Theory of Artificial Satellites to Order Four

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Software programs are designed to normalize the main problem in the theory of artificial satellites. At the onset, a canonical transformation of a new type, called the elimination of the parallax, reduces the system to a quasi-Keplerian one with varying angular momentum. As the first phase in a stepwise refinement, a separable radial intermediary is extracted from the simplified Hamiltonian; its normalization by a Poincaré transformation is executed by machine to the fourth order in closed form, that is to say, without developing the generator in powers of the eccentricity.

Introduction

THERE have been several attempts at developing by computer an analytical solution for the main problem in the theory of an artificial satellite of the Earth. All follow Delaunay's approach to lunar theory. The Hamiltonian of the system is expanded in powers of the eccentricity, the coefficients in the series are trigonometric polynomials in the mean anomaly l and in the argument of the perigee g over the ring of polynomials in the sine of the inclination. On the one hand, the canonical algorithm developed by the author[1] has been reproduced by Kutuzov[2] and then improved by Kinoshita[3] who proceeded therefrom to add the corrections due to the zonal coefficients J_3 and J_4. On the other hand, Challe and Laclaverie[4] chose to integrate immediately the six equations of motion for a special set of elements in a semi-analytical manner; to their series, Berger and Walch[5] added the variations caused by the zonal harmonics from degree three to degree seven.

However, a development in the powers of the eccentricity, while easy to automate by means of a processor of Poisson series, is not the best answer possible. For the solution has been produced in *closed* form by hand to order one by Brouwer[6,7] and to order two by Kozai.[8] The tentatives by Sconzo and Hertz around 1970 to reproduce Kozai's results by FORMAC have not been altogether convincing. A few years later, Henrard announced that he was making progress in his project of checking systematically Kozai's theory[9]; one of his students, H. Claes, has just produced a solution of mixed character,[10] in closed form to order two but expanded in powers of the eccentricity at order three. Our indications are that it is possible to solve the main problem of satellite theory wholly in closed form to order three, perhaps even to order four. Therefore we propose to examine again the problem of expanding by machine a literal solution of the main problem in the theory of artificial satellites.

What prompts us to reopen the question is the discovery that a canonical transformation called the *elimination of the parallax* converts the main problem into a quasi-Keplerian system, i.e. into a two-body problem (satellite-Earth) whose angular momentum is a function of the oblateness in the Earth's gravity field. The removal of the parallax is easy to implement by computer; it puts us in the most favorable position to overcome the major obstacle to a fully automatic expansion of a literal solution of the main problem. Having given up the traditional formats of a development in the powers of the eccentricity, we have to search for a list of variables to appear explicitly at the onset in the reduced quasi-Keplerian approximation; by trials and errors, we have to identify the irrational quantities and irreducible expressions that successive derivations and Poisson brackets will bring into the expressions; we must invent simplification procedures and reductions to standard formats. Indeed Brouwer's artifice for not substituting Delaunay's variables explicitly in the Hamiltonian is a novelty in celestial mechanics; the traditional literature offers no suggestion about selecting explicit variables or simplifying the calculations. Learning to implement Brouwer's suggestion by machine is preparation for its application to major problems like the theories of planets or the lunar theory. The benefits would be considerable, to wit, the application which Hori[11,12] made by hand to Hill's reduced problem.

Programs to predict the positions of a satellite are migrating from large mainframes to minicomputers on the ground or on-board satellites. We are confident that this research will eventually produce the concise analytical solution around which navigation systems responding in real time may be designed.

Elimination of the Node

The dynamical system modelled after the Hamiltonian

$$M = M_{0,0} + \epsilon M_{1,0} \tag{1}$$

$$M_{0,0} = \tfrac{1}{2}(X^2 + Y^2 + Z^2) - \mu/r \tag{2}$$

$$M_{1,0} = -(\mu/r)(\alpha/r)^2 P_2(z/r) \tag{3}$$

is what we name the *main problem* in the theory of artificial satellites. Recall that the physical parameters of the system are 1) the Earth's *Keplerian constant* μ, which is the product of the constant of universal gravitation by the mass of the Earth, 2) the *equatorial radius* α of the Earth, and 3) the coefficient $\epsilon = C_{2,0} = -J_2$ of the zonal harmonic of second degree in the Earth's gravitational potential. The satellite is taken to be a massless particle; its Cartesian coordinates noted (x,y,z) refer to a principal frame of inertia $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ at the barycenter of the Earth; $r = (x^2 + y^2 + z^2)^{1/2}$ is the distance from the center of the Earth to the satellite. The velocity X of the satellite is referred to a geocentric frame fixed in space; its components in the Earth's frame of inertia are noted (X, Y, Z).

Presented as Paper 78-1439 at the AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 7-9 1978; submitted Sept. 1, 1978; revision received Aug. 13, 1980.

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The main problem of artificial satellites constitutes a conservative dynamical system with apparently three degrees of freedom. In fact, the system is invariant with respect to the group of rotations around the polar axis of the Earth, hence it admits another integral beside the energy.[13] The purpose of the following transformation is to bring forth this integral, at the same time using it to reduce the number of degrees of freedom by one unit.

The transformation is not new; it is embedded in the method which Jacobi proposed to find a complete solution of Hamilton's equation for the Keplerian problem. All the same, we name it after Whittaker[14] because he was the first to extricate it from that particular context and to advocate its systematic application in particle dynamics.

One way of defining a Whittaker mapping is by the implicit equations

$$x=\partial S/\partial X \qquad y=\partial S/\partial Y \qquad z=\partial S/\partial Z$$

$$R=\partial S/\partial r \qquad \Theta=\partial S/\partial \theta \qquad N=\partial S/\partial \nu \tag{4}$$

derived from the generating function

$$S\equiv S(X,Y,Z,r,\theta,\nu)=(X\cos\nu+Y\sin\nu)r\cos\theta$$

$$+[(X\sin\nu-Y\cos\nu)^2+Z^2]^{1/2}r\sin\theta$$

It is a canonical transformation of Mathieu type because it satisfies the identity

$$X\mathrm{d}x+Y\mathrm{d}y+Z\mathrm{d}z=R\mathrm{d}r+\Theta\mathrm{d}\theta+N\mathrm{d}\nu$$

Upon using the auxiliary functions

$$c=\cos I=N/\Theta \qquad s=\sin I$$

one may solve the system of equations (4) to find the explicit relations defining the transformation:

$$x=r(\cos\theta\cos\nu-\sin\theta\sin\nu\sin I)$$

$$y=r(\cos\theta\sin\nu+\sin\theta\cos\nu\sin I)$$

$$z=r\sin\theta\cos I$$

$$X=[R\cos\theta-(\Theta/r)\sin\theta]\cos\nu-[R\sin\theta+(\Theta/r)\cos\theta]\sin\nu\sin I$$

$$Y=[R\cos\theta-(\Theta/r)\sin\theta]\sin\nu+[R\sin\theta+(\Theta/r)\cos\theta]\cos\nu\sin I$$

$$Z=[R\sin\theta+(\Theta/r)\cos\theta]\cos I$$

Thus it appears that the Whittaker transformation is a rotation about the center of Earth. Let $\boldsymbol{G}=\boldsymbol{x}\times\boldsymbol{X}$ be the angular momentum of the satellite, and $\boldsymbol{n}$ the unit vector in the direction of $\boldsymbol{G}$. If Θ denotes the norm of the angular momentum, then $\boldsymbol{G}=\Theta\boldsymbol{n}$. The plane normal to $\boldsymbol{n}$ is the satellite's orbital plane; at every instant, it contains the position $\boldsymbol{x}$ and the velocity $\boldsymbol{X}$. The inclination I is determined by the vector relation $\boldsymbol{k}\cdot\boldsymbol{n}=\cos I$. There exists a unit vector $\boldsymbol{l}$ such that $\boldsymbol{k}\times\boldsymbol{n}=\boldsymbol{l}\sin I$; if $I \bmod \pi=0$, then $\boldsymbol{l}$ is an arbitrary unit vector in the equatorial plane $(\boldsymbol{i},\boldsymbol{j})$; otherwise, it is uniquely determined at the intersection of the satellite's orbital plane and the Earth's equatorial plane. The direction $\boldsymbol{l}$ stands for to what astronomers call the *ascending node*. The angles ν and θ satisfy the relations

$$\boldsymbol{l}=\boldsymbol{i}\cos\nu+\boldsymbol{j}\sin\nu=\boldsymbol{x}\cos\theta-(\boldsymbol{k}\times\boldsymbol{x})\sin\theta$$

They are commonly referred to as the longitude of the ascending node and the argument of the latitude. The momentum R conjugate to the coordinate r is the radial velocity, i.e. the projection of the velocity $\boldsymbol{X}$ on the position $\boldsymbol{x}$; N is the polar component of the angular momentum.

The Whittaker transformation converts the terms (2) and (3) of the main problem (1) into the functions

$$M_0\equiv M_0(R,\Theta,r)=\tfrac{1}{2}(R^2+\Theta^2/r^2)-\mu/r \tag{5}$$

$$M_1\equiv M_1(\Theta,N,r,\theta)=(\mu/r)(\alpha/r)^2[\tfrac{1}{2}-\tfrac{3}{4}s^2+\tfrac{3}{4}s^2\cos2\theta] \tag{6}$$

As it is expected from a system invariant with respect to the group of rotations around the polar axis of the Earth, the angle ν is ignorable in the Hamiltonian of the main problem; hence its conjugate momentum N is an integral, precisely the integral predicted by Noether's theorem.

Elimination of the Parallax

The normalization of the Hamiltonian [Eq. (1)] is drastically simplified by means of a canonical transformation to convert the factor $(\mu/r)(\alpha/r)^2$ into $(\mu/p)(\alpha/r)^2$ and to eliminate explicit appearances of the argument of the latitude θ in Eq. (6). The mapping is called the *elimination of the parallax*. The concept is new in celestial mechanics. Its scope is not limited to the main problem of artificial satellite theory; it extends to a wide class of Hamiltonian systems containing, among others, the relativistic corrections, the effects of the Earth's oblateness on the moon's orbit, and the planetary theories.

The elimination of the parallax is a transformation of Lie type.[15,16] It rests on an elementary property of the Lie derivative

$$L_0:F\rightarrow L_0(F)=(F;M_0)$$

in the flow determined by the Keplerian Hamiltonian M_0. Let $F\equiv F(p,C,S,\theta)$ be any function of the semilatus rectum p, the argument of latitude θ and the quantities[17] $C\equiv C(r,\theta,R,\Theta)$ and $S\equiv S(r,\theta,R,\Theta)$ such that

$$\Theta/r-\Theta/p=C\cos\theta+S\sin\theta \qquad R=C\sin\theta-S\cos\theta$$

It is readily seen that

$$(F;M_0)=(\Theta/r^2)\partial_4 F$$

where $\partial_4 F$ designates the coefficient of $\mathrm{d}\theta$ in the gradient

$$\mathrm{d}F=\partial_1 F\mathrm{d}p+\partial_2 F\mathrm{d}C+\partial_3 F\mathrm{d}S+\partial_4 F\mathrm{d}\theta$$

From the standpoint of a perturbation algorithm based on Lie series, the elementary property is interpreted as proving that the quadrature

$$W=\int^{\theta}F(p,C,S,\theta)\,\mathrm{d}\theta$$

is a formal solution of the partial differential equation

$$(W;M_0)=(\Theta/r^2)F$$

The parallax has been eliminated from the main problem [Eq. (2)] to the fourth order in ϵ. The program has been developed and executed in PL/I on an Amdahl 470-6 at the University of Cincinnati. Then it has been rewritten in Fortran to use Dasenbrock's processor of Poisson series[18]; the new code has served at the Naval Research Laboratory on the TI Advanced Scientific Computer to remove the parallax from the main problem to the third order (Alfriend, Dec. 1978) and to the fourth order (Coffey, June 1979). Their results are in complete agreement with those of the author. Considering how different the hardware and software are on both sides, the comparison constitutes a very exacting check.

In the Whittaker chart $(r,\theta,\nu,R,\Theta,N)$ emanating from the Lie transformation, the Hamiltonian (1) becomes the series

$$M=M_{0,0}+\frac{\Theta^2}{r^2}\sum_{n\geq 1}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{0\leq j\leq n/2}e^{2j}\sum_{0\leq k\leq j}M^*_{n,j,k}s^{2k}\cos 2kg \tag{7}$$

The eccentricity and the argument of the perigee are defined in the usual way by the relations

$$p/r=1+e\cos(\theta-g) \qquad R=(\Theta/p)e\sin(\theta-g)$$

the coefficients $M^*_{n,j,k}$ are the polynomials in s^2 listed in Table 1.

Beside simplifying drastically the developments to come in the following section, the elimination of the parallax presents several advantages:

1) The main problem at the *first* order may now be solved in closed form without restriction on either the eccentricity (from 0 to ∞) or the inclination (from 0 to π including the critical inclinations). The solution is short and fit to processing in real time by microprocessors based on an arithmetic chip.[19]

Table 1 Inclination functions for the main problem in the theory of artificial satellites

$$M^*_{1,0,0}=\frac{1}{2}-\frac{3}{4}s^2$$

$$M^*_{2,0,0}=-\frac{5}{4}+\frac{21}{8}s^2-\frac{21}{16}s^4$$

$$M^*_{2,1,0}=-\frac{3}{8}+\frac{3}{8}s^2+\frac{15}{64}s^4$$

$$M^*_{2,1,1}=\frac{21}{16}-\frac{45}{32}s^2$$

$$M^*_{3,0,0}=\frac{39}{4}-\frac{567}{16}s^2+\frac{2961}{64}s^4-\frac{315}{16}s^6$$

$$M^*_{3,1,0}=\frac{87}{16}-\frac{837}{32}s^2+\frac{6813}{128}s^4-\frac{8145}{256}s^6$$

$$M^*_{3,1,1}=-\frac{9}{16}-\frac{117}{32}s^2+\frac{2565}{512}s^4$$

$$M^*_{4,0,0}=-\frac{501}{4}+\frac{18,909}{32}s^2-\frac{131,157}{128}s^4+\frac{50,049}{64}s^6-\frac{13,815}{64}s^8$$

$$M^*_{4,1,0}=-\frac{3633}{32}+\frac{11,961}{32}s^2+\frac{22,509}{256}s^4-\frac{123,309}{128}s^6+\frac{2,596,275}{4096}s^8$$

$$M^*_{4,1,1}=\frac{40,545}{64}-\frac{300,525}{128}s^2+\frac{2,956,191}{1024}s^4-\frac{2,360,115}{2048}s^6$$

$$M^*_{4,2,0}=-\frac{783}{128}+\frac{13,905}{256}s^2-\frac{26,541}{128}s^4+\frac{277,425}{1024}s^6-\frac{1,781,595}{16,384}s^8$$

$$M^*_{4,2,1}=-\frac{567}{32}+\frac{7533}{256}s^2+\frac{50,409}{2048}s^4-\frac{136,215}{4096}s^6$$

$$M^*_{4,2,2}=\frac{37,611}{1024}-\frac{10,665}{128}s^2+\frac{384,345}{8192}s^4$$

2) As long as secular perturbations of the third order are contained, the main problem may even be solved at the second order by removing altogether the perturbations of first and second order. Worthless for global analytical studies because it contains mixed terms in θ, the solution is nonetheless useful for designing and calculating maneuver corrections over a few anomalistic revolutions.[19] The solution is simple enough that it may be evaluated in real time by microprocessors.

3) The general technique of a preparatory canonical transformation offers a method for designing intermediaries leading to concise analytical solutions of the main problem.[20]

4) When the zonal harmonics above the second one are inserted in the main problem, the elimination of the parallax brings a spectacular simplification of the developments; in particular, it leads to an elementary treatment[21] of the second and third order corrections due to J_3-J_7.

The explicit equations for both the direct and inverse transformations resulting from the elimination of the parallax will be published soon.[22]

Normalization of the Intermediary

Removal of the parallax does not solve the main problem; it prepares the next step which is a canonical transformation $(r,\theta,\nu,R,\Theta,N)\rightarrow(l,g,h,L,G,H)$ of Lie type such that: 1) its generator is conditionally periodic in the coordinates l,g,h; 2) it converts the Hamiltonian M into a function depending only on the momenta L,G, and H.

A transformation having these properties is said to *normalize* the main problem. The coordinates (l,g,h) and their conjugate momenta (L,G,H) constitute a set of angle- and action-variables for the dynamical system.

The program which automatically creates a normalization for the main problem is conceived to grow by stepwise refinements. A convenient system to start from is a separable Hamiltonian obtained by omitting a perturbation from the main problem. Actually, after the parallax has been eliminated, the Hamiltonian given in Eq. (7) may be restructured as the sum

$$M=R_2+\epsilon^2P$$

$$R_2=M_{0,0}+\frac{\Theta^2}{r^2}\sum_{n\geq 1}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{0\leq j\leq n/2}e^{2j}M^*_{n,j,0}$$

$$P=\frac{\Theta^2}{r^2}\sum_{n\geq 1}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{1\leq j\leq n/2}e^{2j}\sum_{1\leq k\leq j}M^*_{n,j,k}s^{2k}\cos 2kg \tag{8}$$

The Hamiltonian R_2 is invariant with respect to the group of rotations about the center of the Earth in the satellite's orbital plane. Indeed the argument of latitude θ is ignorable in R_2; its conjugate momentum Θ is the integral predicted by Noether's theorem. Incidentally, the perturbation P is not invariant for that group and Θ is not an integral for the main problem itself.

In accordance with Liouville's theorem, since it has three integrals in involution (Θ, N, and R_2 itself), the dynamical system represented by R_2 is integrable. Considering also that the perturbation is of the second order in ϵ, there follows that R_2 is a *natural intermediary* for the main problem.

With only one effective degree of freedom, namely in the radial elements r and R, the intermediary R_2 is separable.

For all these reasons, it has been decided that the program to normalize the full main problem will be derived by augmenting the program which normalizes the separable intermediary R_2 with provisions to take care of the special situations caused by the terms in $\cos 2g$ and $\cos 4g$ present in the perturbation P.

In the best conditions, the computer algorithms should be tested on a separable Hamiltonian whose normal form may be obtained by hand. By the same token, we would rather begin with a dynamical system simple enough to make it relatively inexpensive to pursue the normalization beyond the second order, at least to order three, if not to order four. It is not that satellite's observations warrant that much accuracy, but the transformation at the fourth order is bound to tell something about the conditions under which Brouwer's artifice for eschewing an explicit substitution of Delaunay's variables may be transfered to more exacting systems like the lunar and planetary theories.

The terms in e^2 present in the Hamiltonian R_2 preclude a simple way of reaching a general solution either in the Whittaker or in the Delaunay chart. By omitting the terms in e^2 for a while, we find in the radial intermediary

$$R_I = M_{0,0} + \frac{\Theta^2}{r^2} \sum_{n \geq 1} \frac{\epsilon^n}{n!} \left(\frac{\alpha}{p}\right)^{2n} M^*_{n,0,0} \tag{9}$$

a good starter to program the normalization. The reason is that Eq. (9) defines a quasi-Keplerian system

$$R_I = ½ (R^2 + \Theta'^2/r^2) - \mu/r \tag{10}$$

Its angular momentum $\Theta' \equiv \Theta'(\Theta, s^2, \epsilon)$ is determined by the relation

$$\Theta'^2 = \Theta^2 \sum_{n \geq 1} \frac{\epsilon^n}{n!} \left(\frac{\alpha}{p}\right)^{2n} M^*_{n,0,0}$$

The canonical transformation, called a *torsion,* defined by the implicit equations

$$\theta = \partial T/\partial\Theta \qquad \nu = \partial T/\partial N \qquad \Theta' = \partial T/\partial\theta' \qquad N' = \partial T/\partial\nu' \tag{11}$$

from the generator

$$T \equiv T(\theta', \nu', \Theta, N) = \nu' N + \theta'\Theta' = \nu' N$$
$$+ \theta'\Theta \left[1 + \sum_{n \geq 1} \frac{\epsilon^n}{n!} \left(\frac{\alpha}{p}\right)^{2n} M^*_{n,0,0} \right]^{½}$$

converts the Hamiltonian (9) into the Keplerian system (10). Interestingly enough, the torsion is an extension of the geometric construction outlined by Newton[23]—and dubbed by Cotes the *revolution of orbits*—to map the Hamiltonian system

$$H = ½ (R^2 + \Theta^2/r^2) - \mu/r + k^2/r^2 \tag{12}$$

onto a Keplerian Hamiltonian.

Equations (11) of the torsion can be made explicit by using an extension of Lagrange's Inversion Formula.[24] All calculations have been executed by hand to order four in ϵ by Mrs. Deprit-Bartholomé. The results make the foundations of an extensive dictionary of formulas against which the results of the normalization have been checked.

Unfortunately, the classical solution of the main problem may not be recovered by extending the elementary torsion in Eq. (11) to accommodate the terms in e^2 brought by R_2 and the terms in cos2g and cos4g contained in the perturbation P. In fact, Brouwer resorts to a general perturbation algorithm devised by Poincaré to remove the angles l and g from the main problem. Our task is thus to develop the program which executes Poincaré's normalization on the elementary intermediary R_I.

Brouwer and Kozai compute averages over the mean anomaly l by taking quadratures over l. Considering that the integrands are not expressed explicitly as trigonometric polynomials in l, averaging by quadratures is an awkward operation to automate in this instance. Instead, we propose to relay on differential identities based on the properties of the Lie derivative in a Keplerian flow. In the case of R_I, we need only the identity

$$(\phi/G; M_0) + \eta^3/p^2 = 1/r^2 \tag{13}$$

for the equation of the center $\phi = f - l$ to remove the short period term $1/r^2$ together with the general class of trigonometric identities

$$[(\sin jf)/Gj; M_0] = (\cos jf)/r^2 \tag{14}$$

to remove the terms which are periodic in the true anomaly f.

In Eq. (13), η is the dimensionless quantity $(1 - e^2)^{½}$. On account of Eqs. (13) and (14), the partial differential equations occurring at the end of each order to determine the generator W_n of the transformation and the normalized term $M_{0,n}$ in the intermediary will be solved formally *without evaluating quadratures* merely by inspecting terms. We read in Eq. (14) that a term in $(\cos jf)/r^2$ makes no contribution to the normalized intermediary whereas it generates a term in $(\sin jf)$ for the generator. Likewise, Eq. (13) implies that a term in $1/r^2$ produces a term in $1/p^2$ for the normalized intermediary and a term in ϕ for the generator of the Lie transformation.

The major source of difficulties in the automation of the normalization resides in the partial derivatives. After a number of unfortunate attempts, all ending in a sudden explosion of some intermediate result, a successful strategy has been developed. All terms entering the calculations must be kept in standard formats dictated by their physical dimensions:

1) a function having the dimension of an energy, if it is already normalized, must be maintained in the form

$$A \equiv A(L, G, H) = (G^2/p^2)(\alpha/p)^{2n} \sum_j A_j \eta^j \tag{15}$$

the coefficients A_j being polynomials in s^2 over the field of rational integers. Save for the generic term (G^2/p^2) that has the dimension of an energy, the factors in the product are dimensionless.

2) a function having the dimension of an energy, if it is not yet normalized, must be reduced to the form

$$P \equiv P(r, f, R, \Theta, N)$$

$$= (\Theta^2/r^2)(\alpha/p)^{2n} \sum_j e^j (\cos jf) \sum_k (P_{j,k}\eta^k + P^*_{j,k}\beta^k) \tag{16}$$

the coefficients $P_{j,k}$ and $P^*_{j,k}$ being polynomials in s^2 over the field of rational integers. The generic factor (Θ^2/r^2) carries the physical dimensions of an energy while the other terms are dimensionless.

3) a function having the dimension of an angular momentum will be reduced to the form

$$W = G(\alpha/p)^{2n} \Big[\phi \sum_k (W_{0,k}\eta^k + W^*_{0,k}\beta^k)$$
$$+ \sum_j e^j (\sin jf) \sum_k (W_{j,k}\eta^k + W^*_{j,k}\beta^k) \Big] \tag{17}$$

the coefficients $W_{0,k}$, $W^*_{0,k}$, $W_{j,k}$ and $W^*_{j,k}$ being polynomials in s^2 over the field of rational integers. The generic factor G supports the dimensions which are that of an action or angular momentum.

In Eqs. (16) and (17), β stands for the dimensionless quantity $(1 + \eta)^{-1}$.

From the standard formats [Eqs. (15-17)] there result three rules of simplification:

1) Unwanted powers in $1/r$ are removed by multiplying the nonstandard factor a sufficient number of times by the

"parallactic identity"

$$1 = r(1+e\cos f)/p$$

2) It can be shown that the energy functions which are the elements of the Lie triangle for the normalization as well as the generators of the transformation present the d'Alembert characteristic[7,25] for the pair (e,f). The standard formats [Eqs. (16) and (17)] stipulate that the eccentricity appears only as a factor of $\cos jf$ or $\sin jf$, that it must appear whenever a term in $\cos jf$ or $\sin jf$ appears, and that it does so with an exponent equal to the multiple of the true anomaly. Excessive powers of e^2 are removed by multiplication with the identity

$$1 = (1-\eta^2)/e^2$$

3) Products in $\beta^j\eta^k$ and $\beta^j\eta^{-k}(k>0)$ are automatically decomposed in sums of powers of β alone and of powers of η alone by means of the identities

$$1 = (1-\beta)/\beta\eta \quad \text{or} \quad 1 = (1-\beta\eta)/\beta$$

It remains now to plan how the partial derivatives should be chained to produce the Poisson brackets in the standard format [Eqs. (15) and (16)]. This is the place where an overlooked simplification usually results in an explosive proliferation of intermediary terms which the simplification rules are unable to reduce. Poisson brackets are invariant with respect to the group of canonical transformations. Therefore we are at liberty to choose the coordinate set with respect to which the partial derivatives in a Poisson bracket are performed. If the bracket involves an energy term that is already normalized, the partial derivatives are taken in the Delaunay chart; otherwise, they are taken in the Whittaker chart.

In the Whittaker chart, the partial derivatives of an energy-like term E are to be computed as follows:

$$\frac{r}{2}\frac{\partial E}{\partial r} = -E + \frac{\partial E}{\partial e}\left(\frac{r}{2}\frac{\partial E}{\partial r}\right) \qquad \frac{G}{p}\frac{\partial E}{\partial R} = \frac{\partial E}{\partial e}\left(\frac{G}{p}\frac{\partial e}{\partial R}\right)$$

Table 2 Inclination functions for the normalized intermediary R_1

$$R_{1,1} = \frac{1}{2} - \frac{3}{4}s^2$$

$$R_{2,1} = -\frac{3}{2} + \frac{27}{8}s^2 - \frac{15}{8}s^4$$

$$R_{2,2} = -\frac{3}{4} + \frac{9}{4}s^2 - \frac{27}{16}s^4$$

$$R_{3,1} = \frac{197}{16} - \frac{351}{8}s^2 + \frac{3267}{64}s^4 - \frac{765}{32}s^6$$

$$R_{3,2} = \frac{27}{4} - \frac{405}{16}s^2 + \frac{999}{32}s^4 - \frac{405}{32}s^6$$

$$R_{3,3} = \frac{3}{2} - \frac{27}{4}s^2 + \frac{81}{8}s^4 - \frac{81}{16}s^6$$

$$R_{4,1} = -156 + \frac{23{,}841}{32}s^2 - \frac{169{,}047}{128}s^4 + \frac{16{,}605}{16}s^6 - \frac{2385}{8}s^8$$

$$R_{4,2} = -\frac{369}{4} + \frac{3699}{8}s^2 - \frac{56{,}835}{64}s^4 + \frac{49{,}113}{64}s^6 - \frac{15{,}795}{64}s^8$$

$$R_{4,3} = -27 + \frac{567}{4}s^2 - \frac{1107}{4}s^4 + \frac{3807}{16}s^6 - \frac{1215}{16}s^8$$

$$R_{4,4} = -\frac{15}{4} + \frac{45}{2}s^2 - \frac{405}{8}s^4 + \frac{405}{8}s^6 - \frac{1215}{64}s^8$$

The factors affecting the partial derivatives are arranged so that quantities on the left side of the relations have the dimensions of an energy while the terms in the right side are the products of an energy by dimensionless factors. Likewise, for the generator, which is an action, the partial derivatives are calculated by the rules:

$$\frac{2}{r}\frac{\partial W}{\partial R} = \left(\frac{1}{G}\frac{\partial W}{\partial e}\right)\left(\frac{2G}{r}\frac{\partial W}{\partial R}\right) \qquad \frac{p}{G}\frac{\partial W}{\partial r} = \left(\frac{1}{G}\frac{\partial W}{\partial e}\right)\left(p\frac{\partial W}{\partial r}\right)$$

Left- and right-hand members are dimensionless.

Derivations with respect to the eccentricity require great care. It is a matter of treating each class of terms separately, of preparing a number of particular cases and of organizing the differentiation procedures around tables of decision. Details concerning the eccentricity functions and their derivatives in the Whittaker chart will be given in a note[26] dealing with the normalization of the full radial intermediary R_2.

The intermediary R_1 comes out of the normalization as the series

$$R_1 = \tfrac{1}{2}n^2a^2 + n^2a^2\sum_{n\geq 1}\frac{\epsilon^n}{n!}\left(\frac{\alpha}{p}\right)^{2n}\sum_{1\leq j\leq n}R_{n,j}\eta^j$$

the coefficients $R_{n,j}$ are the polynomials in s^2 listed in Table 2.

The generator of the Lie transformation which normalizes the radial intermediary R_1 is simple. We present here only the first two terms:

$$W_1 = G(\alpha/p)^2\phi(\tfrac{1}{2} - \tfrac{3}{4}s^2)$$

$$W_2 = -G\left(\frac{\alpha}{p}\right)^4\left\{\phi\left(\frac{3}{2} + \frac{27}{8}s^2 - \frac{15}{8}s^4\right)\right.$$

$$\left. + \beta\left[\left(\frac{1}{2} - \frac{3}{2}s^2 + \frac{9}{8}s^4\right)e\sin f + \left(\frac{1}{8} - \frac{3}{8}s^2 + \frac{9}{32}s^4\right)e^2\sin 2f\right]\right\}$$

Comparing W_2 with the generator S_2 in the von Zeipel's transformation derived by Kozai suggests immediately how significantly the elimination of the parallax simplifies the solution of the main problem in the theory of artificial satellites.

Table 3 Profile of the normalization

	Elimination of the parallax	Normalization of intermediary R_1
List operations		
Garbage collection	8296	992
Link trigonometric nodes	113,483	11,815
Monomial nodes	56,795	23,837
Coefficient nodes	121,394	24,648
Compose trigonometric nodes	71,934	1837
Monomial nodes	23,970	17,503
Decompose trigonometric nodes	72,166	5236
Monomial nodes	38,478	16,360
Rational arithmetic		
Add/subtract	102,162	20,957
Halve	35,794	820
Multiply/divide	64,486	5378
Poisson series algebra		
Add/subtract	2876	155
Multiply	2768	95
Differentiate	2740	30
Atom/molecule		55

Fortunately, the normalization of the rudimentary radial intermediary R_I is self-checking: in the generator W, at every order, the coefficient of ϕ depends neither on η nor on β, it being merely a polynomial in s^2. The programs do not make use of this property; they only register that all terms in η and β that contribute to the coefficient of ϕ happen to cancel one another exactly to zero.

The two programs built thus far have maintained a "profiler" in line, i.e. a count of how many times a primitive of the processor of Poisson series has been entered.[27] The summaries in Table 3 of the counts made by the profiler give an accurate idea of how complex the programs are. Hardly any work is necessary at order one, most of the calculations have been done by hand at order two; the complications arise at the third order and the task becomes heavy at order four.

The budget is by all means encouraging.

The fourth order in the theory of artificial satellites is a luxury. It is pursued for two reasons: from a mathematical standpoint, to elucidate the structures of the phase flow in the main problem, and from a computational standpoint, to test how Brouwer's artifice could be applied to lunar and planetary theories.

To an analytical development of the third order, one might object that, in an operational context, evaluation of long trigonometric polynomials in several arguments is not a practical alternative to numerical integration. But the interface between symbolic calculations and numerical evaluations is a problem that has received a great deal of attention in the past five years.[28] Tehniques are now available not only to evaluate multiple Fourier series as quickly as possible, but also to automatically convert a Poisson series into an optimal subroutine to evaluate the series.[29]

Conclusions

Whatever complexities are encountered at the third order, the fact remains that, after the parallax has been removed, the main problem in the theory of artificial satellites becomes trivial at the first order and very tame at the second order.

The present paper is an updated and slightly extended revision of the one given at the AIAA/AAS Astrodynamics Conference held in Palo Alto in August of 1978. Much progress has been realized in the mean time. Basically the present author and Dr. Coffey have solved the main problem in closed form to order four, but there will be some time before the results and the techniques are documented well enough to be submitted for publication. Conceptually, the ideas expressed here are leading to a revision of the concept of a Delaunay normalization. It can be expressed in a form invariant with respect to the group of canonical transformations, namely as a Birkhoff normalization relative to the Keplerian Hamiltonian. From there one can devise new normalizing algorithms which are considerably shorter than the traditional ones. Now that most experts in celestial mechanics know how to automate their algebraic calculations by computer, the time has come to face the more difficult problem of generating shorter series which adhere more closely to the physical situations they represent.

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